

Proof of the holographic formula for entanglement entropy

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ABSTRACT: Entanglement entropy for a spatial partition of a quantum system is studied in theories which admit a dual description in terms of the anti-de Sitter (AdS) gravity one dimension higher. A general proof of the holographic formula which relates the entropy to the area of a codimension 2 minimal hypersurface embedded in the bulk AdS space is given. The entanglement entropy is determined by a partition function which is defined as a path integral over Riemannian AdS geometries with non-trivial boundary conditions. The topology of the Riemannian spaces puts restrictions on the choice of the minimal hypersurface for a given boundary conditions. The entanglement entropy is also considered in Randall-Sundrum braneworld models where its asymptotic expansion is derived when the curvature radius of the brane is much larger than the AdS radius. Special attention is paid to the geometrical structure of anomalous terms in the entropy in four dimensions. Modification of the holographic formula by the higher curvature terms in the bulk is briefly discussed.

KEYWORDS: AdS-CFT Correspondence, Large Extra Dimensions.

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1. Introduction

Entanglement entropy measures a degree of the correlation between different subsystems of a quantum system. In many-body theories the quantum entanglement is usually determined when subsystems are spatially separated. A considerable attention has been paid in last few years to studying entanglement entropy in 2D spin-chain models near the critical point with the purpose to understand better collective effects in strongly correlated systems (for a review of some of these results see [1]).

In a similar way, one could pose the question about entanglement on the fundamental level, in a quantum gravity theory. It was conjectured in [2] that for partition of the ground state by a plane the entanglement entropy per unit area of the plane is

$$\mathcal{S} = \frac{1}{4G_N} , \quad (1.1)$$

where G_N is the gravitational constant in the low-energy gravity sector of the theory. \mathcal{S} measures entanglement of all genuine microscopical degrees of freedom of the fundamental theory.

Relation (1.1) has been motivated by the black hole physics:¹ the density of the Bekenstein-Hawking entropy per unit area of the black hole horizon is universal and equals $1/(4G_N)$. The fact that quantum entanglement and gravity are connected phenomena is interesting because one can study properties of the gravitational couplings in "gravity analogs", by using entanglement entropy in condensed matter systems [2].

¹First indications on a possible relation between quantum entanglement and gravity phenomena should be attributed to authors of [3–5] who suggested to interpret the Bekenstein-Hawking entropy as the entanglement entropy.

If the fundamental theory lives in a space-time with number of dimensions higher than four there is an issue of higher-dimensional extension of (1.1). There may be at least two options. In theories of the Kaluza-Klein (KK) type all degrees of freedom propagate in the bulk. In this case partition of all KK modes by a hypersurface \mathcal{B} in four-dimensions is equivalent to partition of higher-dimensional fields by the surface $\tilde{\mathcal{B}} = \mathcal{B} \times \Omega_{D-4}$ where Ω_{D-4} is a small compact space of $D - 4$ extra-dimensions. If (1.1) is true the entanglement entropy should be

$$S = \frac{\mathcal{A}}{4G_N} = \frac{\tilde{\mathcal{A}}}{4G_N^{(D)}} \quad , \quad (1.2)$$

where \mathcal{A} is the area of \mathcal{B} and $\tilde{\mathcal{A}} = \mathcal{A} \cdot \text{vol } \Omega_{D-4}$ is the volume of $\tilde{\mathcal{B}}$. Equation (1.2) takes into account the relation $G_N^{(D)} = G_N \cdot \text{vol } \Omega_{D-4}$, between higher-dimensional, $G_N^{(D)}$, and low-dimensional, G_N , gravitational couplings.

The other option is the theories with large extra dimensions. In these theories the gravity remains higher-dimensional while matter fields are confined on a brane and do not propagate in the bulk. For this reason, the way how the separating surface is extended in extra-dimensions should be determined by the bulk gravity equations.

There is a particular example where this issue can be investigated: the Randall-Sundrum model where the bulk is the anti-de Sitter (AdS) gravity. If \mathcal{B} is a separating hypersurface in the brane theory, its extension $\tilde{\mathcal{B}}$ has to be a minimal surface embedded in the AdS space. The boundary of $\tilde{\mathcal{B}}$ on the brane is \mathcal{B} . With this prescription the entropy in the brane theory takes the form (1.2),

$$S = \frac{\tilde{\mathcal{A}}}{4G_N^{(d+1)}} \quad , \quad (1.3)$$

where $G_N^{(d+1)}$ is the gravitational constant in the bulk and d is the number of dimensions on the brane.

Equation (1.3) is the "holographic formula" which was first suggested by Ryu and Takayanagi [7, 8] for the entropy of conformal field theories (CFT) which admit a dual description in terms of the AdS gravity one dimension higher. The remarkable property of the result of [7, 8] is that (1.3) enables one to study quantum entanglement in strongly correlated systems, such as strongly coupled gauge theories, where the direct field theoretical (QFT) computations for the entropy are extremely difficult.

Ryu and Takayanagi presented an intuitive derivation of (1.3) by using AdS/CFT correspondence [9]–[11] and provided some examples in its support. The analysis of [7, 8] was restricted by static problems and $\tilde{\mathcal{B}}$ was assumed to be a static minimal surface in a Lorenzian AdS background. In [12] equation (1.3) was used to argue that the entropy of a black hole in a braneworld model in three dimensions is an entanglement entropy. In [13] implications of (1.3) were considered for the entropy of a de Sitter brane.

The main purpose of our work is to understand better where and at which conditions relation (1.3) comes from. The present paper is organized as follows. The proof of the holographic formula is given in section 2. For a finite-temperature theory the entanglement entropy can be derived in a statistical mechanical manner from a partition function

$Z^{CFT}(\beta, T)$ of the boundary CFT. $Z^{CFT}(\beta, T)$ is defined on a class of Riemannian manifolds \mathcal{M}_n (manifolds with the Euclidean signature). The parameter $\beta = 2\pi n$, where n is a natural number, can be interpreted as an inverse "temperature". By using the AdS/CFT conjecture we write $Z^{CFT}(\beta, T)$ as a path integral in the bulk AdS gravity with the boundary condition that the conformal boundary of "histories" $\tilde{\mathcal{M}}_n$ involved in the path integral belongs to the conformal class of \mathcal{M}_n . The bulk spaces have conical singularities on codimension 2 hypersurfaces $\tilde{\mathcal{B}}$ which are "holographically dual" to the separating surface \mathcal{B} in the boundary CFT. Evaluation of the path integral in the saddle point approximation picks up a space $\tilde{\mathcal{M}}_n$ where the classical AdS gravity action has an extremum. The variational principle for the action implies that $\tilde{\mathcal{B}}$ has to be a minimal surface embedded in a Riemannian AdS background. It then immediately follows that in the saddle point approximation the entanglement entropy is given by (1.3). The derivation of (1.3) answers some questions left open in [7, 8] and makes the statement more precise. For instance, the topology of a Riemannian space puts restrictions on the choice of the minimal hypersurface under chosen boundary conditions. Connection of the entanglement entropy with the path integral in AdS gravity enables one to study entanglement in the theories which are stationary but not static and to consider theories with different phases.

In section 3 we study consequences of (1.3) in RS brane-world scenario. In particular, we derive the asymptotic geometrical structure of the entropy when the curvature radius of the brane is much larger than the AdS radius. In section 4 a general form of the "anomalous term" in the entropy is given in four-dimensional brane theories, including the case when \mathcal{B} has non-vanishing extrinsic curvatures. Modification of the holographic formula by the higher curvature terms in the bulk action is briefly analyzed in section 5. We discuss our results in section 6. It is emphasized that if the corrections caused by the curvature of the brane are neglected the area density of the ground state entanglement entropy in the theory on the brane is given by (1.1) where G_N is the Newton coupling in the gravity induced on the brane, in complete agreement with the suggestion of [2].

2. Holographic formula for the entanglement entropy

An extended form of the AdS/CFT conjecture [9]–[11], which we follow in this paper, asserts that a supergravity theory in the anti-de Sitter space is dual to a conformal field theory on the boundary of that region. The correspondence takes the following form²

$$Z^{CFT}[h] = \int [Dg] \exp(-I_{\text{gr}}[g]) \quad . \quad (2.1)$$

Equation (2.1) relates the partition function $Z^{CFT}[h]$ of a CFT ($\ln Z^{CFT}[h]$ is a generating functional of connected Green's functions) on a manifold \mathcal{M} with a metric $h_{\mu\nu}$ to the gravity theory in a space-time one dimension higher. The path integral in the right hand

²We consider Euclidean theory. We also assume that the bulk space has $d+1$ dimensions. All geometrical structures in the bulk are denoted with a tilde. The corresponding "dual" quantities on the boundary will be denoted by the same letters without the tilde. For example, the bulk Riemannian manifold $\tilde{\mathcal{M}}$ corresponds to a boundary manifold \mathcal{M} .

side (r.h.s.) in (2.1) is taken over all geometries which induce a given conformal class of metrics $h_{\mu\nu}$ at their conformal boundary. We denote bulk manifolds and metrics as $\tilde{\mathcal{M}}$ and g_{KL} , respectively. The action in (2.1) is the gravity action

$$I_{\text{gr}} = -\frac{1}{16\pi G_N^{(d+1)}} \int_{\tilde{\mathcal{M}}} d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{l^2} \right) - \frac{1}{8\pi G_N^{(d+1)}} \int_{\partial\tilde{\mathcal{M}}} d^d x \sqrt{h} K \quad (2.2)$$

with the negative cosmological constant $\Lambda = -d(d-1)/(2l^2)$. To avoid the volume divergences of I_{gr} one considers a cut of the bulk manifold at some d -dimensional hypersurface $\partial\tilde{\mathcal{M}}$ at some large radius. In the limit of infinite radius $\partial\tilde{\mathcal{M}}$ belongs to the conformal class of \mathcal{M} . The boundary term in (2.2), which depends on the metric $h_{\mu\nu}$ induced on $\partial\tilde{\mathcal{M}}$ and on the trace of its extrinsic curvature K , is the Gibbons-Hawking term which is necessary for a well-defined variational problem. The corresponding conformal theory on $\partial\tilde{\mathcal{M}}$ has an ultraviolet cutoff related to the radius of the boundary.

In analogy with (2.1) we formulate the method of computing entanglement entropy S in the boundary CFT in terms of the path integral over AdS metrics. We start with a recipe of computing S in a QFT described in detail in [2].

Consider a QFT model of a field variable ϕ defined on a $(d-1)$ dimensional hypersurface Σ . The state of the system, can be described in general by the density matrix $\hat{\rho}$ and in what follows we assume that it is a state at temperature T . The ground state can be obtained in the limit of zero temperature. When applying our results to conformal theories which have AdS duals, Σ can be, for example, a hyperplane R^{d-1} , a hypersphere S^{d-1} , or a Lobachevsky hyperbolic space H^{d-1} , see [14]. In these cases \mathcal{M} is $R^{d-1} \times S^1$, $S^{d-1} \times S^1$, or $H^{d-1} \times S^1$. There may be more complicated situations which we mention below.

Suppose that Σ is divided onto two parts, Σ_1 and Σ_2 by a codimension 2 hypersurface \mathcal{B} and define a reduced density matrix for the region Σ_1 ,

$$\hat{\rho}_1 = \text{Tr}_2 \hat{\rho} \ ,$$

by taking trace over the states located in the region Σ_2 . The entanglement entropy is defined as the von Neumann entropy

$$S_1 = -\text{Tr}_1 \hat{\rho}_1 \ln \hat{\rho}_1 \ . \quad (2.3)$$

Analogously, one can define the entanglement entropy S_2 in the region Σ_2 by tracing the density matrix over the states located in the region Σ_1 . If the system is in a pure state $|\psi\rangle$, i.e. $\hat{\rho} = |\psi\rangle\langle\psi|$, it is not difficult to show that $S_1 = S_2$, see [3]. In what follows we do all our computations for the entropy in Σ_1 . To simplify the notations we write the entropy and all associated objects without subscript 1. Distinctions in the entropies for different regions will be explained below.

Let us introduce a discrete parameter $\beta = 2\pi n$, and define a "partition function"

$$Z(\beta, T) = \int [D\tilde{\phi}] e^{-I[\phi, \beta, T]} \ , \quad (2.4)$$

where $I[\phi, \beta, T]$ is the classical action for the field ϕ defined on a Riemannian manifold \mathcal{M}_n .

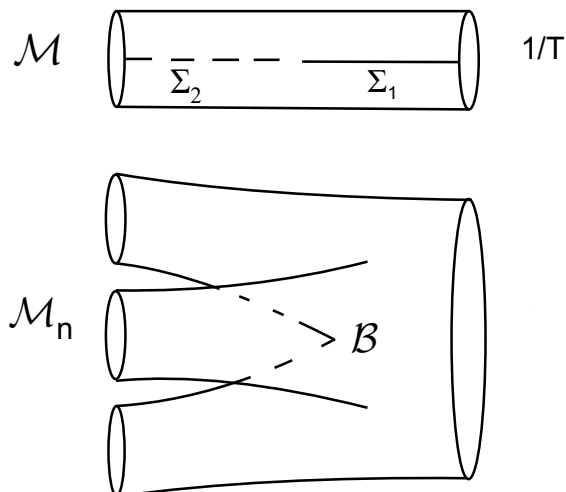


Figure 1: Spaces \mathcal{M} and \mathcal{M}_n (for $n = 3$) are shown schematically on upper and lower pictures. The theory at temperature T is defined on a space $\Sigma_1 \cup \Sigma_2$ shown as an interval.

The definition of \mathcal{M}_n and $Z(\beta, T)$ is as follows. If $n = 1$ (2.4) coincides with the partition function of the given field ϕ in space Σ at finite temperature T . The background manifold in this case is $\mathcal{M}_1 = \Sigma \times S^1 \equiv \mathcal{M}$, where S^1 is the circle of the length T^{-1} . To obtain \mathcal{M}_n for $n > 1$ one has to take n copies of \mathcal{M} each with a cut along Σ_1 and glue them along the cuts as is shown on figure 1. The entanglement entropy is given by the equation [2]

$$S(T) = - \lim_{\beta \rightarrow 2\pi} \left(\beta \frac{\partial}{\partial \beta} - 1 \right) \ln Z(\beta, T) \quad . \quad (2.5)$$

The operation with the parameter β in (2.5) should be understood in the following way: one first computes $Z(\beta, T)$ for $\beta = 2\pi n$, and then replaces β with a continuous parameter. This can be done even if \mathcal{M}_n itself cannot be defined at arbitrary β . It will be important for us that (2.5) coincides with the formula for the entropy in statistical mechanics if $Z(\beta, T)$ is interpreted as a partition function and β^{-1} as a temperature³.

In the same way, by cutting n copies of \mathcal{M} and gluing along Σ_2 one can obtain the entanglement entropy in the region Σ_2 .

Let us emphasize that definition (2.5) can be also applied to the entropy in theories which are stationary but not static. To give an example, suppose that Σ has an axial isometry generated by a Killing vector field ∂_φ . One can study the entanglement entropy in the frame of reference which rigidly rotates with an angular velocity $\dot{\varphi} = \Omega$. The space \mathcal{M} is a torus which is obtained by identifications $(\tau, \varphi) \sim (\tau, \varphi + 2\pi)$ and $(\tau, \varphi) \sim (\tau + 1/T, \varphi + \Omega/T)$ (i.e., along the trajectories generated by ∂_φ and $\partial_\tau + \Omega\partial_\varphi$). Gluing n copies of \mathcal{M} along the cut Σ_1 yields \mathcal{M}_n . Computation of the entanglement entropy in this setting has to be accompanied by the change Ω to $i\Omega$ in the final results.

³The parameter β^{-1} should not be confused with the physical temperature T .

Equation (2.5) is a convenient starting point for computing the entropy in a CFT by using the bulk AdS gravity. To this aim we treat $Z^{CFT}(\beta, T)$ as a genuine partition function but taken for a particular class of background spaces. Then relation (2.1) enables one to represent it as

$$Z^{CFT}(\beta, T) = \int [Dg] \exp(-I_{\text{gr}}[g]) \quad . \quad (2.6)$$

The path integral in the right hand side in (2.6) is taken over all $(d + 1)$ dimensional geometries $\tilde{\mathcal{M}}_n$ whose asymptotic boundary is related to \mathcal{M}_n by a conformal transformation. Like in (2.1) formula (2.6) implies a spatial cutoff in the bulk.

Description of "histories" in (2.6) is parallel to the construction of \mathcal{M}_n . First, one considers all AdS geometries whose asymptotic boundaries are from the conformal class of \mathcal{M} . We denote these geometries $\tilde{\mathcal{M}}$. Then, one makes different cuts of $\tilde{\mathcal{M}}$ along d -dimensional hypersurfaces $\tilde{\Sigma}_1$. One requires that the asymptotic boundary of $\tilde{\Sigma}_1$ is conformal to the cut Σ_1 in \mathcal{M} . This boundary condition does not fix $\tilde{\Sigma}_1$ uniquely. There may be different cuts of $\tilde{\mathcal{M}}$ which have this property. Finally, by taking n identical copies of $\tilde{\mathcal{M}}$ with the same cut $\tilde{\Sigma}_1$ and gluing them along the cuts one gets a space $\tilde{\mathcal{M}}_n$ with the required boundary condition.

Integral (2.6) can be computed in the saddle point approximation. To do the computation one has to take into account that $\tilde{\mathcal{M}}_n$ have conical singularities which lie on $(d - 1)$ -dimensional hypersurfaces $\tilde{\mathcal{B}}$ where all n cuts meet. The surplus of the conical angle around each point of $\tilde{\mathcal{B}}$ is $\beta - 2\pi = 2\pi(n - 1)$. The classical action (2.2) in this case should be

$$I_{\text{gr}} = -\frac{1}{16\pi G_N^{(d+1)}} \int_{\tilde{\mathcal{M}}_n/\tilde{\mathcal{B}}} d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{l^2} \right) - \frac{1}{8\pi G_N^{(d+1)}} \int_{\partial\tilde{\mathcal{M}}_n} d^d x \sqrt{h} K - \frac{1}{8\pi G_N^{(d+1)}} (2\pi - \beta) \tilde{\mathcal{A}} \quad . \quad (2.7)$$

The integral in the r.h.s. of (2.7) goes over the regular region $\tilde{\mathcal{M}}_n/\tilde{\mathcal{B}}$ of $\tilde{\mathcal{M}}_n$. The delta-function like singularities in the curvature R at $\tilde{\mathcal{B}}$ result in the last term in the r.h.s. of (2.7) where $\tilde{\mathcal{A}}$ is the volume of $\tilde{\mathcal{B}}$.

As in (2.2) one considers a cut of the bulk manifold at some d -dimensional hypersurface $\partial\tilde{\mathcal{M}}_n$ to avoid the volume divergences. The space $\partial\tilde{\mathcal{M}}_n$ has conical singularities and one might worry that they yield additional boundary terms. The structure of possible boundary terms could be

$$\frac{1}{G_N^{(d+1)}} \int_{\partial\tilde{\mathcal{B}}} d^{d-1}x \sqrt{\sigma} f(\beta, \mathcal{R}, \mathcal{K}, \dots)$$

where the integral goes over $(d - 1)$ -dimensional hypersurface $\partial\tilde{\mathcal{B}}$, a location of singular points on $\partial\tilde{\mathcal{M}}$, with the determinant of the metric σ . Function $f(\beta, \mathcal{R}, \mathcal{K}, \dots)$ should be a local invariant of all possible combinations of internal, \mathcal{R} , and external, \mathcal{K} , characteristics of $\partial\tilde{\mathcal{B}}$. The problem is that f has to have the dimension of the length, while the invariants have the dimension of the inverse length in some non-negative power. Note also that one cannot use the AdS radius l to construct combinations with required dimensionality

because the form of boundary terms (for off-shell metrics) does not depend on the presence of the cosmological constant in the bulk. Thus, we conclude that conical singularities do not generate boundary terms.

In the saddle point approximation (2.6) yields

$$\ln Z^{CFT}(\beta, T) \simeq -I_{\text{gr}}[\bar{g}] \quad , \quad (2.8)$$

where the bulk metric \bar{g} is a point where (2.7) has an extremum,

$$\delta I_{\text{gr}}[\bar{g}] = 0 \quad (2.9)$$

under the given boundary conditions. Equation (2.8) holds in the case when there is a dominating contribution from one of extremal configurations, so that other contributions can be neglected. We will assume that this is the case. Because one has to consider variational procedure at a fixed value of β one gets from (2.7), (2.9) two sorts of equations: the standard Einstein equations in the bulk and equation for $\tilde{\mathcal{B}}$ which ensures that the variation of the last term in the r.h.s. of (2.7) vanishes,

$$R_{KL} - \frac{1}{2}\bar{g}_{KL}R + \Lambda\bar{g}_{KL} = 0 \quad , \quad (2.10)$$

$$\delta\tilde{\mathcal{A}} = 0 \quad . \quad (2.11)$$

Condition (2.11) results in the Nambu-Goto equations which describe embedding equations of $\tilde{\mathcal{B}}$ as a minimal hypersurface in bulk space $\tilde{\mathcal{M}}_n$. The metric \bar{g} of $\tilde{\mathcal{M}}_n$ is a solution to (2.10). The boundary condition for (2.11) requires that asymptotic conformal boundary of $\tilde{\mathcal{B}}$ belongs to the conformal class of \mathcal{B} (which is the separating surface of Σ and the place of location of conical singularities of \mathcal{M}_n). It should be noted that spaces $\tilde{\mathcal{M}}_n$ and $\tilde{\mathcal{M}}$ are identical outside the conical singularities. Therefore, $\tilde{\mathcal{B}}$ should be also a minimal hypersurface in $\tilde{\mathcal{M}}$.

One can use now (2.5) and (2.7) to get holographic formula (1.3) for the entanglement entropy in a CFT theory

$$\begin{aligned} S^{CFT}(T) &= - \lim_{\beta \rightarrow 2\pi} \left(\beta \frac{\partial}{\partial \beta} - 1 \right) \ln Z^{CFT}(\beta, T) \\ &\simeq \frac{\tilde{\mathcal{A}}}{4G_N^{(d+1)}} \quad . \end{aligned} \quad (2.12)$$

Let us summarize conditions for (2.12).

1) The formula holds in the semiclassical approximation when contribution of the bulk gravitons (quantum fluctuations of the bulk geometry) is neglected. See discussion of this point in section 5.

2) A single dominating "trajectory", \bar{g} , in the path integral (2.6) which solves bulk equations (2.10) is implied.

3) \bar{g} has the positive signature. Thus, the minimal surface $\tilde{\mathcal{B}}$ is embedded in the Riemannian AdS background \bar{g} . If the boundary space \mathcal{M} is static $\tilde{\mathcal{B}}$ is also a minimal static surface in the Lorenzian section of \bar{g} , the case initially considered in [7, 8].

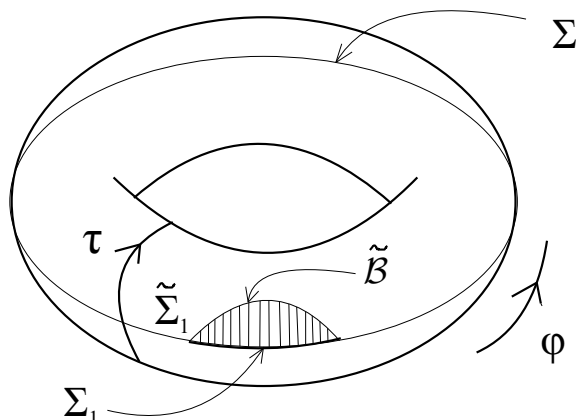


Figure 2: A Euclidean BTZ black hole $\tilde{\mathcal{M}}$ is shown as a solid torus. The shaded region is the hypersurface $\tilde{\Sigma}_1$ with the boundary Σ_1 . It lies inside the torus. The manifold $\tilde{\mathcal{M}}_n$ corresponding to the BTZ black hole is obtained by cutting and gluing n copies of $\tilde{\mathcal{M}}$ along $\tilde{\Sigma}_1$.

4) By the construction, the choice of $\tilde{\mathcal{B}}$ implies that there is a cut $\tilde{\Sigma}_1$ inside the bulk space $\tilde{\mathcal{M}}$ such that the boundary $\partial\tilde{\Sigma}_1$ consists of $\tilde{\mathcal{B}}$ and a conformal infinity which belongs to the conformal class of Σ_1 . The boundary $\partial\tilde{\Sigma}_1$ does not have other elements.

The last condition enables one to understand why in general the entanglement entropy in the region Σ_1 is different from the entanglement in its completion Σ_2 . This happens when there are no cuts in $\tilde{\mathcal{M}}$ connecting Σ_2 to $\tilde{\mathcal{B}}$.

The latter situation occurs at finite temperatures for topological reasons. Let us illustrate it by using AdS_3/CFT_2 . If the CFT_2 is at non-zero temperature T its gravity dual is a BTZ black hole [15] given by the metric

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{l^2}{r^2 - r_+^2}dr^2 + r^2d\varphi^2 = r_+^2 \sinh^2(\rho/l)d\tau^2 + d\rho^2 + r_+^2 \cosh^2(\rho/l)d\varphi^2, \quad (2.13)$$

where $r = r_+ \cosh(\rho/l)$, $0 \leq \tau \leq 2\pi l/r_+$, $0 \leq \varphi \leq 2\pi$. The Euclidean horizon is located at $r = r_+$ or $\rho = 0$. The Euclidean BTZ black hole has the topology of a solid torus, as is shown in figure 2.

To get rid off the volume divergences we assume that $0 \leq \rho \leq \rho_0$, where ρ_0 is a cutoff. The surface Σ is a circle $\rho = \rho_0$, $\tau = 0$, while Σ_1 is a segment of this circle as is shown on figure 2. It can be demonstrated that constant τ geodesics which start and end at the circle $\rho = \rho_0$, $\tau = 0$ make a turn before reaching the horizon $\rho = 0$. One of such geodesics, $\tilde{\mathcal{B}}_1$, restricts the cut $\tilde{\Sigma}_1$ of the torus. The cut starts at Σ_1 and lies in the half-plane $\tau = 0$. The other geodesic line, $\tilde{\mathcal{B}}_2$, which connects the ends of Σ_1 should be associated to the completion Σ_2 , see figure 3.

The partition function $Z^{CFT}(\beta, T)$ of the CFT in the semiclassical approximation is given by the classical action (2.8) on a space $\tilde{\mathcal{M}}_n$ which is obtained by cutting along $\tilde{\Sigma}_1$ and gluing n copies of the torus.

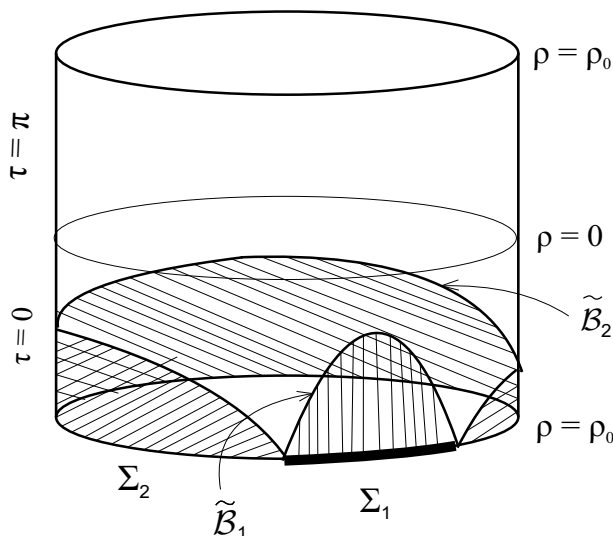


Figure 3: A slice by a plane of the torus shown on figure 2. The Euclidean horizon $\rho = 0$ lies on the slice. The shaded regions are the hypersurfaces $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$. Their boundaries inside the slices, $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$, are the geodesic lines.

The cut of the torus $\tilde{\Sigma}_1$ which starts at Σ_1 and the cut $\tilde{\Sigma}_2$ which starts at Σ_2 end at different geodesics, $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$, respectively. Any cut which starts at Σ_2 and ends at \mathcal{B}_1 has to additionally cross the surface of the torus⁴. Such cuts violate the boundary conditions and are prohibited.

3. Entanglement entropy in brane worlds

Consider now the braneworld models [6] where a d -dimensional gravity is induced on a brane embedded in a $(d + 1)$ -dimensional AdS spacetime. The (Euclidean) RS model has the following action⁵

$$\begin{aligned}
 I_{\text{RS}} = & -\frac{1}{16\pi G_N^{(d+1)}} \int_{\tilde{\mathcal{M}}} d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{l^2} \right) \\
 & - \frac{1}{8\pi G_N^{(d+1)}} \int_{\partial\tilde{\mathcal{M}}} d^d x \sqrt{h} K + I_{\text{brane}} \quad .
 \end{aligned}
 \tag{3.1}$$

Here I_{brane} is the action of the brane. It consists of a contribution depending on the brane tension $\mu = (d - 1)/(4\pi G_N^{(d+1)} l)$ and a contribution I_{matter} of matter fields localized on the brane.

The action (3.1) is finite because the brane is located at a finite radius r . Conformal infinity of AdS corresponds to the limit $r \rightarrow 0$. At small r the functional I_{RS} takes the

⁴It is easy to understand that this happens because Σ cannot be contracted inside the torus to a point.

⁵We consider the model with a single brane which separates two copies of the AdS spaces. The copies are cut at the location of the brane $\mathcal{M} = \partial\tilde{\mathcal{M}}$ and glued together. It is implied that the integrals in (3.1) go over the two copies of AdS spaces.

following form (provided that the bulk metric obeys (2.10)):

$$I_{\text{RS}} = I_{\text{matter}} + \Gamma_{\text{RS}} + I_{(1)} + I_{(2)} + \dots \quad (3.2)$$

The action is reduced to a non-local part Γ_{RS} and a series of terms $I_{(k)}$ which are integrals of k -th power polynomials of the curvatures of \mathcal{M} and their derivatives. The first two terms are [14, 16]–[18]

$$I_{(1)} = -\frac{1}{16\pi G_N^{(d)}} \int_{\mathcal{M}} d^d x \sqrt{h} R \quad , \quad (3.3)$$

$$I_{(2)} = -\frac{l^3}{16\pi G_N^{(d+1)}(d-2)^2(d-4)} \int_{\mathcal{M}} d^d x \sqrt{h} \left(R^{\mu\nu} R_{\mu\nu} - \frac{d}{4(d-1)} R^2 \right) \quad , \quad (3.4)$$

where R and $R_{\mu\nu}$ are the Ricci scalar and the Ricci tensor of the boundary metric, and

$$G_N^{(d)} = \frac{(d-2)G_N^{(d+1)}}{2l} \quad . \quad (3.5)$$

It is implied in (3.4) that $d > 4$. In $d = 4$ one has to replace the factor $1/(d-4)$ in $I_{(2)}$ by $\ln(r/r_0)$, where r_0 is some renormalization length scale.

From the point of view of the AdS/CFT correspondence the non-local piece, Γ_{RS} , can be interpreted as a "renormalized" part of the effective action for a CFT on the brane [19], while local terms $I_{(k)}$ are associated to the ultraviolet divergences regularized in the presence of the cutoff r . Therefore, the brane theory (in the absence of matter on the brane, $I_{\text{matter}} = 0$) is a RS CFT plus gravity theory. The gravity on the brane is an induced phenomenon, in a sense close to the Sakharov's induced gravity scenario [20].

The gravitational coupling $G_N^{(d)}$ can be expressed in terms of the microscopical parameters of the CFT theory. If $d = 4$ the CFT is a $U(N)$ superconformal field theory. The CFT parameters and the supergravity parameters are related by

$$\frac{l^3}{G_N^{(5)}} = \frac{2N^2}{\pi} \quad . \quad (3.6)$$

To compute the entanglement entropy in the theory on the brane one can proceed as in section 2. Let us suppose that geometry is $\mathcal{M} = \Sigma \times S^1$ where S^1 has circumference length T^{-1} . We are interested in the entanglement entropy $S^{RS}(T)$ associated with a part Σ_1 of Σ . The entropy can be computed with the help of the formula

$$S^{RS}(T) = -\lim_{\beta \rightarrow 2\pi} \left(\beta \frac{\partial}{\partial \beta} - 1 \right) \ln Z^{RS}(\beta, T) \quad . \quad (3.7)$$

At $\beta = 2\pi n$ quantity $Z^{RS}(\beta, T)$ is the partition function of the brane theory on a manifold \mathcal{M}_n which is obtained by gluing n copies of \mathcal{M} along the cut made along Σ_1 . By using AdS/CFT interpretation of the RS theory one can identify $Z^{RS}(\beta, T)$ to the path integral in the bulk AdS gravity with the condition that \mathcal{M}_n coincides with the boundary hypersurface embedded in the bulk manifolds $\tilde{\mathcal{M}}_n$ at a finite radius. In the semiclassical approximation

$$\ln Z^{RS}(\beta, T) \simeq -I_{\text{RS}}(\beta, T) \quad . \quad (3.8)$$

The classical action, I_{RS} , has the structure (2.7) where the bulk metric of $\tilde{\mathcal{M}}_n$ solves (2.10) and the induced boundary metric solves the boundary Israel equations,

$$-\frac{1}{4\pi G_N^{(d+1)}}(K_{\mu\nu} - h_{\mu\nu}K) = \mu h_{\mu\nu} + T_{\mu\nu} \quad , \quad (3.9)$$

where $T_{\mu\nu}$ is the stress energy tensor of matter fields on the brane. The bulk and boundary equations are defined outside the location of conical singularities where \mathcal{M}_n locally coincides with \mathcal{M} . Thus, these equations are the same as equations in the standard RS setting. The variational procedure in presence of conical singularities requires that the singularities in the each copy of bulk AdS are on a minimal hypersurface $\tilde{\mathcal{B}}$ obeying (2.11). The intersection of $\tilde{\mathcal{B}}$ with the boundary is the hypersurface \mathcal{B} which divides Σ to Σ_1 and Σ_2 . Therefore, in the semiclassical approximation

$$S^{RS}(T) \simeq \frac{\tilde{\mathcal{A}}}{4G_N^{(d+1)}} \quad , \quad (3.10)$$

where $\tilde{\mathcal{A}}$ is the doubled volume of $\tilde{\mathcal{B}}$. By using representation of the RS action (3.2) one can also write the entropy in another form

$$S^{RS}(T) = S_{\text{TH}}(T) + \frac{\mathcal{A}}{4G_N^{(d)}} + S_{(2)} + \dots \quad (3.11)$$

Here \mathcal{A} is the volume of \mathcal{B} and $S_{\text{TH}}(T)$ is the entanglement entropy in the "renormalized" RS CFT,

$$S_{\text{TH}}(T) = -\lim_{\beta \rightarrow 2\pi} \left(\beta \frac{\partial}{\partial \beta} - 1 \right) \ln \Gamma_{\text{RS}}(\beta, T) \quad . \quad (3.12)$$

The term in (3.11) proportional to \mathcal{A} appears from $I_{(1)}$. Corrections $S_{(k)}$ in (3.11) correspond to $I_{(k)}$ with $k \geq 1$.

Note that the only piece in the entropy (3.11) which depends both on the quantum state and on the hypersurface Σ_1 , where the entangled states are located, is a "thermal part" $S_{\text{TH}}(T)$. Other terms in $S^{RS}(T)$ have a pure geometrical structure and can be interpreted as a "vacuum part". The "vacuum parts" of the entropies in Σ_1 and in its completion Σ_2 coincide.

The thermal part $S_{\text{TH}}(T)$ dominates over the vacuum part in the high temperature regime. It can be shown [7, 8] by using properties of minimal surfaces in the AdS that in this limit $S_{\text{TH}}(T)$ is proportional to the volume of Σ_1 , in agreement with the extensive properties of the entanglement entropy in quantum field theories at high temperatures [2].

4. Higher order corrections and conformal anomaly

Let us dwell on the structure of terms $S_{(k)}$ in (3.12) which are determined by functionals $I_{(k)}$,

$$S_{(k)} = \lim_{\beta \rightarrow 2\pi} \left(\beta \frac{\partial}{\partial \beta} - 1 \right) I_{(k)} \quad . \quad (4.1)$$

The first term $I_{(1)}$ yields $S_{(1)} = \mathcal{A}/(4G_N^{(d)})$, see (3.11). When the separating hypersurface \mathcal{B} is a smooth closed manifold isometrically embedded in \mathcal{M} and extrinsic curvatures of \mathcal{B} vanish one gets (for $d > 4$)

$$S_{(2)} = \frac{l^3}{4G_N^{(d+1)}(d-2)^2(d-4)} \int_{\mathcal{B}} d^{d-2}x \sqrt{\sigma} \left(\mathcal{R}_{ii} - \frac{d}{2(d-1)}(\mathcal{R} + 2\mathcal{R}_{ii} - \mathcal{R}_{ijij}) \right) \quad , \quad (4.2)$$

where summation over indexes $i, j = 1, 2$ is implied. The quantity \mathcal{R} is the Ricci scalar of \mathcal{B} . Other two invariants are defined in terms of the Ricci and Riemann tensors of \mathcal{B} , $\mathcal{R}_{ii} = n_i^\mu n_i^\nu \mathcal{R}_{\mu\nu}$, $\mathcal{R}_{ijij} = n_i^\mu n_j^\nu n_i^\lambda n_j^\rho \mathcal{R}_{\mu\lambda\nu\rho}$, where n_i^μ are two unit vectors in \mathcal{M} orthogonal to \mathcal{B} and normalized, $(n_i, n_j) = \delta_{ij}$.

Derivation of (4.2) is as follows. One keeps in mind that expressions (3.2)–(3.4) are valid when the RS action is taken on a smooth solution of the bulk, (2.10), and boundary, (3.9), equations. Manifolds \mathcal{M}_n which appear under computation of the RS partition function $Z^{RS}(\beta, T)$ have conical singularities and cannot be considered. Let us replace \mathcal{M}_n by a smooth manifold which differs from \mathcal{M}_n only in some narrow domain around \mathcal{B} where conical singularities are "regularized" by some method (smoothing of conical singularities is described in [21]). The "regularized" bulk manifold $\tilde{\mathcal{M}}_n$ is defined as a solution to bulk equations (2.10) with the condition that its boundary is the regularized \mathcal{M}_n . When regularization is removed the brane geometry approaches \mathcal{M}_n while bulk space tends to $\tilde{\mathcal{M}}_n$. The RS action on the smoothed manifolds has the asymptotic form (3.2) where $I_{(k)}$ are well defined. As was explained in [21], if one removes the regularization and then takes the limit $\beta = 2\pi$ in (4.1) the quantities $S_{(k)}$ will be finite and will not depend on the regularization procedure⁶. In particular, the results of [21] for the polynomials quadratic in curvatures can be used to get (4.2).

A special interest is the entropy in four dimensions, $d = 4$. In this case (4.2) has to be modified as follows

$$S_{(2)} = -\frac{l^3}{48G_N^{(5)}} \ln \frac{r}{r_0} \int_{\mathcal{B}} d^2x \sqrt{\sigma} (2\mathcal{R} + \mathcal{R}_{ii} - 2\mathcal{R}_{ijij}) \quad , \quad (4.3)$$

where the combination $l^3/G_N^{(5)}$ is given in (3.6). Because of the logarithmic factor this term breaks the scaling invariance of the CFT. Explicit connection of $r dS_{(2)}/dr$ to the trace anomaly of the stress energy tensor has been found in [8]. The structure of the anomalous piece discussed in [8] coincides with (4.3).

Till now we assumed that the separation surface \mathcal{B} has vanishing extrinsic curvatures. This happens in the case when \mathcal{M} has a Killing vector field and \mathcal{B} is the set of fixed points of this field. It is the case where results of [21] are applied. The contribution of extrinsic curvatures in $S_{(2)}$ can be established in $d = 4$ by using arguments pointed out in [22]. Because the integral of the anomalous trace of the stress energy tensor is invariant under the conformal transformations $S_{(2)}$ in this dimension should have the same property. One

⁶Note that powers of the curvature are not well-defined when the regularization is removed. That is why one cannot use this method in deriving from (3.2) gravity theory induced on the brane with conical singularities.

has to focus on the transformation of two last terms in the r.h.s. of (4.3). The changes of the metric $h_{\mu\nu} \rightarrow e^{-2\omega} h_{\mu\nu}$ result in the change

$$\mathcal{R}_{ii} - 2\mathcal{R}_{ijij} \rightarrow \mathcal{R}_{ii} - 2\mathcal{R}_{ijij} - 2n_i^\mu n_i^\nu (\nabla_\mu \nabla_\nu \omega + \nabla_\mu \omega \nabla_\nu \omega) \quad , \quad (4.4)$$

On the other hand, the extrinsic curvatures transform as

$$k_{\mu\nu}^i \rightarrow e^{-\omega} (k_{\mu\nu}^i + (n^i)^\lambda \omega_{,\lambda} \sigma_{\mu\nu}) \quad ,$$

where $\sigma_{\mu\nu} = h_{\mu\nu} - n_\mu^i n_\nu^i$. Therefore, the conformally invariant generalization of (4.3) in the presence of non-zero extrinsic curvatures should be [22]

$$S_{(2)} = -\frac{l^3}{48G_N^{(5)}} \ln \frac{r}{r_0} \int_{\mathcal{B}} d^2x \sqrt{\sigma} \left[2\mathcal{R} + \mathcal{R}_{ii} - 2\mathcal{R}_{ijij} + \frac{1}{2}(k^i)^2 + \lambda ((k^i)^2 - 2\text{Tr}(k^i k^i)) \right] \quad , \quad (4.5)$$

where $k^i = (k^i)^\mu_\mu$, $\text{Tr}(k^i k^i) = (k^i)_{\mu\nu} (k^i)^{\mu\nu}$. The combination $(k^i)^2 - 2\text{Tr}(k^i k^i)$ is conformally invariant and the constant λ has to be determined by a different method.

Let us also note that \mathcal{B} was assumed to be a smooth closed manifold in Σ . If \mathcal{B} has boundaries there will be extra contributions to $S_{(k)}$. Calculations of the entanglement entropy in QFT's taking into account boundary effects can be found in [2].

5. Effect of higher curvature terms in the bulk

Quantum effects in the bulk AdS space may modify holographic formula (1.3). One type of these modifications is the appearance of higher curvature terms in the bulk action (2.2). Let us consider a simple example when the higher curvature term added to the r.h.s. of (2.2) is the Gauss-Bonnet (GB) term ⁷,

$$I_{GB} = -\frac{\alpha}{16\pi G_N^{(d+1)}} \int_{\tilde{\mathcal{M}}} d^{d+1}x \sqrt{g} (R^2 - 4R_{KL}R^{KL} + R_{KLMN}R^{KLMN}) + I_{GB,b} \quad . \quad (5.1)$$

Here α is some coefficient with the dimension of length square and $I_{GB,b}$ is the corresponding boundary term. The derivation of entanglement entropy in the modified theory is the same as in section 2.

To find the entropy one has to consider the bulk action on spaces $\tilde{\mathcal{M}}_n$ which have conical singularities on a codimension 2 hypersurface $\tilde{\mathcal{B}}$, a holographic dual of the corresponding separating surface \mathcal{B} . The bulk gravity action with the GB term can be written as

$$I_{gr} = I'_{gr} + (\beta - 2\pi)\mathcal{I} \quad , \quad (5.2)$$

where I'_{gr} is the action functional on the regular domain $\tilde{\mathcal{M}}_n/\tilde{\mathcal{B}}$ of $\tilde{\mathcal{M}}_n$ and ⁸

$$\mathcal{I} = \frac{1}{8\pi G_N^{(d+1)}} \int_{\tilde{\mathcal{B}}} d^{d-1}x \sqrt{\tilde{\sigma}} (1 + 2\alpha \tilde{\mathcal{R}}) \quad . \quad (5.3)$$

⁷The gravitational action with the GB term appears in the low-energy string models [23].

⁸To find contribution of the conical singularities in (5.1) we used results of [21]. We also assumed for simplicity that extrinsic curvatures of $\tilde{\mathcal{B}}$ vanish. Possible effects of conical singularities in $I_{GB,b}$ in (4.1) as well as boundary terms in (5.3) are ignored.

Here $\tilde{\sigma}$ is the determinant of metric tensor $\tilde{\sigma}_{ab}$ of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{R}}$ is the Ricci scalar of $\tilde{\mathcal{B}}$. The entanglement entropy in the semiclassical approximation is

$$S^{CFT} = 2\pi\mathcal{I} = \frac{\tilde{\mathcal{A}}}{4G_N^{(d+1)}} + \frac{\alpha}{2G_N^{(d+1)}} \int_{\tilde{\mathcal{B}}} d^{d-1}x \sqrt{\tilde{\sigma}} \tilde{\mathcal{R}} \quad (5.4)$$

which is the result of application (2.5) to (5.2). Thus, the Gauss-Bonnet term yields a correction to holographic formula (1.3) which is proportional to the integral curvature of $\tilde{\mathcal{B}}$. Modification of (1.3) by the GB term has been also discussed in [13].

The bulk metric \bar{g}_{KL} now has to be a solution to equations (2.10) modified by GB terms. Equations which determine position of $\tilde{\mathcal{B}}$ correspond to an extremum of functional (5.3), $\delta\mathcal{I} = 0$. Let ζ^a , $a = 1, \dots, d-1$ be some coordinates on $\tilde{\mathcal{B}}$. The embedding of $\tilde{\mathcal{B}}$ is described by equations $X^K = X^K(\zeta)$. Variations of X^K in \mathcal{I} on the given bulk metric \bar{g}_{KL} yield

$$T^{ab} \left(X^K_{||ab} + \bar{G}_{LM}^K X^L_{,a} X^M_{,b} \right) + T^{ab}{}_{||a} X^K_{,b} = 0 \quad , \quad (5.5)$$

$$T^{ab} = \sigma^{ab} + 4\alpha \left(\frac{1}{2} \sigma^{ab} \tilde{\mathcal{R}} - \tilde{\mathcal{R}}^{ab} \right) \quad . \quad (5.6)$$

Here $\tilde{\sigma}^{ab}$ is the inverse matrix of the metric $\tilde{\sigma}_{ab} = \bar{g}_{KL} X^K_{,a} X^L_{,b}$ induced on $\tilde{\mathcal{B}}$, symbol $||$ denotes covariant derivatives on $\tilde{\mathcal{B}}$, and \bar{G}_{LM}^K are the connections for the bulk metric. If $\alpha = 0$ equations (5.5) reduce to the standard Nambu-Goto equations.

6. Discussion

The main purpose of our work was to give a proof of the holographic formula (1.3) for the entanglement entropy in QFT's which have a dual description in terms of the AdS gravity. In section 2 we have formulated conditions under which (1.3) can be applied. In particular, we demonstrated that $\tilde{\mathcal{B}}$, a holographic dual of the separating surface \mathcal{B} , has to be embedded in a Riemannian AdS space. We also pointed out topological obstructions on the choice of $\tilde{\mathcal{B}}$. Our analysis incorporates static problems discussed in [7, 8], as well as the case of static black holes on the brane when the separating surface is associated to the black hole horizon [12]. It also allows one to extend computations in stationary but non-static theories along the lines explained in section 2.

Let us return to formula (1.1) and compare it with result (3.11) obtained in the RS braneworld model. Suppose that the braneworld geometry is flat, $\mathcal{M} = R^d$. Suppose also that the system is in the ground state and the separating surface is a plane. In this case terms $S_{\text{TH}}(T)$ and $S_{(k)}$ for $k \geq 2$ vanish ($S_{(k)}$ are some invariants constructed in terms of extrinsic and intrinsic geometrical characteristics of \mathcal{B}). The density of entanglement entropy \mathcal{S}^{RS} per unit area of \mathcal{B} becomes

$$\mathcal{S}^{RS} = \frac{1}{4G_N^{(d)}} \quad , \quad (6.1)$$

in agreement with (1.1). In general there should be corrections to (6.1) when \mathcal{M} and \mathcal{B} are not flat.

According to (6.1) quantum entanglement of the degrees of freedom of the fundamental gravity theory can be measured in terms of the gravitational coupling $G_N^{(d)}$ on the brane. There are two ways to describe these degrees of freedom. At energies below the scale l^{-1} these degrees of freedom are the field variables of the brane CFT with the UV cutoff at l^{-1} . In four dimensions the gravity induced by such a CFT has the coupling $G_N^{(d)} \sim l^2/N^2$, which agrees with eq. (3.5), (3.6). At the energies above l^{-1} the brane CFT is itself an effective theory. The genuine degrees of freedom are related to the string theory which provides an exact expression $G_N^{(4)} = \pi l^2/(2N^2)$, where $l = \lambda^{1/4} l_s$, $\lambda = g_{YM}^2 N$ and l_s is the string length.

We do not give a direct microscopical counting of the entanglement entropy on the brane. Our analysis is suggestive. However, the described picture is very similar to that in condensed matter models. Consider the ground state entanglement entropy S in the spin chains near the critical point [1], for instance, in the Ising model. Here the entropy can be directly derived from the density matrix determined by the Ising Hamiltonian. Near the critical point the Ising model corresponds to a 2D QFT with two massive fermion fields. This QFT at the critical point becomes a conformal theory. Thus, the alternative derivation of S can be given in terms of the effective QFT with the UV cutoff associated with the inverse lattice spacing. The two derivations yield the same result for S , see the details in [1].

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